# Conformance Testing of Mealy Machines Under Input Restrictions 

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#### Abstract

We introduce a grey-box conformance testing method for networks of interconnected Mealy Machines. This approach addresses the scenario where all interfaces of the component under test are observable, but its inputs are under the control of other white-box components. We prove new conditions for full fault detection that exploit repetitions across branching executions of the composite machine in a novel way. Finally, we provide experimental evaluation of our approach on cascade compositions of up to a thousand states, and show that it notably outperforms existing black-box testing techniques.


## 1 Introduction/motivation

In this paper we propose a grey-box testing approach for networks of interconnected Mealy Machines. We address the scenario where all communications of the component under test can be observed, but some of its inputs are controlled by other white-box parts of the system. The presented method falls within the scope of conformance testing of finite state machines (FSMs) 3,6

In its most studied variant, the conformance testing problem for FSMs deals with deterministic and input-complete FSMs, i.e., Mealy machines $15,25,20$, 8, 7, 22. In this setting, we consider a fully known Mealy machine $M$ (the specification) and a black box $B$, for which we only know a bound $k$ on the number of states. The goal is to design a test suite to determine whether the black box $B$ conforms (is equivalent) to the $M$.

FSM-based conformance testing is an active research area and numerous techniques exist in the literature (see the survey [6], or 24]). The primary motivation of these techniques is the verification of reactive systems for which FSMs are a suitable model. Despite its simplicity, the FSM formalism is used in very diverse domains, yielding a broad range of applications for FSM-based testing [3]. Another notable application of conformance testing lies in automata learning 5 and derived procedures, such as black-box checking [16. In the setting of the "minimally adequate teacher" introduced by Angluin [1], such techniques require an equivalence oracle in their application. However, these oracles are largely impossible to obtain when dealing with black box systems. Thus, in practice equivalence queries are simulated through various kinds of testing strategies [10]. Furthermore, there is a well-known close relation between model inference and conformance testing (see 2$]$ ) that extends to even more recent automata learning techniques that do not require equivalence oracles [23, 26].

In reality however, reactive systems rarely consist of a single monolithic structure, but instead consist of smaller interacting components. Existing FSM-based techniques developed for black-box systems are not fit to deal with this context, as they suffer from the problem of state explosion and rapidly hit a wall. Hence, there is a need for gray-box methods able to exploit information about known internal components and their communications. There are a few notable conformance testing works in this direction $\sqrt[19]{19}, 18,17$, but it remains a relatively unexplored area.

We consider a scenario where all interfaces of the component under test $B$ are observable, but its inputs are controlled by other known components of the system. The prototypical example of this occurs when $B$ is the tail component of a cascade composition of Mealy machines $B \circ H$, as depicted in Figure 3. The State-Counting method [17], one of the main approaches for this situation, resorts to treating $B$ as a partially specified Mealy machine- i.e., a machine where some transitions are missing. This reduction relies on a classical construction for component minimization by Kim and Newborn 12 which involves an exponential blow-up of the problem's size. However, it has been shown recently that this expensive construction is not required to optimize components 14, and that cheaper techniques may be used instead.

Our main contribution in this paper is a generalization of the State-Counting method which avoids the Kim-Newborn construction. In order to achieve this, we develop a formalism for reasoning about interleaving executions in systems with universal branching. This allows us to prove new sufficient conditions for complete fault-detection in the gray-box setting. We give two testing algorithms making use of this newly introduced theory, and show experimentally that they are able to handle compositions of up to a thousand states, whereas experimental data on reasonably sized examples does not exist for the state-of-the-art 19 , 17. Additionally, we show a practical relation between the gray-box testing task and the classical problem of determining language inclusion between non deterministic automata (NFA) [13] as well as the problem of state reduction for NFAs 11.

## 2 Preliminaries

General Notation Given an alphabet $X$, we write $X^{*}$ for the set of finite words of arbitrary length over $X$. We use $\epsilon$ to denote the empty word, and given a word $\alpha,|\alpha|$ stands for its length. We write $(\alpha<\beta) \alpha \leq \beta$ when $\alpha$ is a (strict) prefix of $\beta$.

Automata Over Finite Words We consider automata over finite words where all states are accepting. Let $\varphi$ be a finite alphabet. A non-deterministic finite automaton (NFA) $A$ over $\varphi$, is a tuple ( $\varphi, S_{A}, \Delta_{A}, r_{A}$ ), where $S_{A}$ is a finite set of states, $\Delta_{A}: S_{A} \times \varphi \rightarrow 2^{S_{A}}$ is the transition function, and $r_{A} \in S_{A}$ is the initial state. A run of $A$ on a word $\alpha \in \varphi^{*}$ is defined as usual. We say that an state $s \in S_{A}$ accepts a word $\alpha$ if there is a run of $A$ on $\alpha$ starting from $s$. If
$s=r_{A}$ we simply say that $A$ accepts $\alpha$. The language of $s$ is the set $\mathcal{L}_{A}(s) \subseteq \varphi^{*}$ containing the words accepted by $s$. Note that $\mathcal{L}_{A}(s)$ is prefix-closed. We simply write $\mathcal{L}_{A}$ for $\mathcal{L}_{A}\left(r_{A}\right)$. We lift $\Delta_{A}$ to words $\alpha \in \varphi^{*}$ in the natural way. The set $\Delta_{A}(s, \alpha)$ consists of all $s^{\prime}$ such that some run of $A$ on $\alpha$ from $s$ finishes at $s^{\prime}$. We write $\Delta_{A}(\alpha)$ for $\Delta_{A}\left(r_{A}, \alpha\right)$.

Mealy Machines A Mealy machine $M$ is a tuple ( $I_{M}, O_{M}, S_{M}, \delta_{M}, \lambda_{M}, r_{M}$ ), where $I_{M}, O_{M}$ are finite alphabets, $S_{M}$ is a finite set of states, $\delta_{M}: S_{M} \times I_{M} \rightarrow$ $S_{M}$ is the next state function, $\lambda_{M}: S_{M} \times I_{M} \rightarrow O_{M}$ is the output function and $r_{M} \in S_{M}$ is the initial state. We lift $\delta_{M}$ and $\lambda_{M}$ to input sequences in the natural way. We define $\delta_{M}(s, \epsilon)=s, \lambda_{M}(s, \epsilon)=\epsilon$ for all $s$. Given $\alpha \in I_{M}^{*}, x \in I_{M}$, if $s^{\prime}=\delta_{M}(s, \alpha)$, then $\delta_{M}(s, \alpha x)=\delta_{M}\left(s^{\prime}, x\right)$ and $\lambda_{M}(s, \alpha x)=\lambda_{M}(s, \alpha) \lambda_{M}\left(s^{\prime}, x\right)$. We write $\delta_{M}(\alpha)$ and $\lambda_{M}(\alpha)$ for $\delta_{M}\left(r_{M}, \alpha\right)$ and $\lambda_{M}\left(r_{M}, \alpha\right)$ respectively. We say that $M$ is reduced if for any pair of different states $s_{1}, s_{2} \in S_{M}$ there is a word $\alpha \in I_{M}$ distinguishing them, i.e., $\lambda_{M}\left(s_{1}, \alpha\right) \neq \lambda_{M}\left(s_{2}, \alpha\right)$. We define $\operatorname{Out}(M)$ as the set of words $\lambda_{M}(\alpha)$, for all $\alpha \in I_{M}^{*}$.

### 2.1 Conformance Testing

Let $M$ be a Mealy machine representing an intended model or specification for a black-box system $B$. A test suite for $M$ is a finite prefix-closed set $E \subseteq\left(I_{M}\right)^{*}$. Sequences $\alpha \in\left(I_{M}\right)^{*}$ are called tests. We define suites as prefixclosed sets, because it simplifies the exposition of technical results later on. However, in practice only the maximal tests in a suite $E$ are relevant. This is because once the output response $\lambda_{B}(\alpha)$ of $B$ to a test $\alpha$ is observed, the outputs $\lambda_{B}(\beta)$ for all $\beta \leq \alpha$ are known as well. Thus, we define the total length, or the number of symbols of a suite $E$ as the sum of the lengths of its maximal tests.

We denote by $\Im$ the set of Mealy machines $N$ with the same input/output alphabets as $M$, and write $\Im_{k}$ for the set of those with at most $k$ states. Given a machine $N \in \Im$, and a set $V \subseteq\left(I_{M}\right)^{*}$, we write $M \sim_{V} N$ if $\lambda_{M}(\alpha)=\lambda_{N}(\alpha)$ for all $\alpha \in V$, or simply write $M \sim N$ when $V=\left(I_{M}\right)^{*}$. We say that a suite $E$ is $k$-complete if $M \sim_{E} N$ implies $M \sim N$ for all $N \in \Im_{k}$. The conformance testing problem for Mealy machines is as follows.

Problem 1 (Unrestricted conformance testing). Given a Mealy machine $M$ and a number $k \in \mathbb{N}$, compute a $k$-complete suite $E$ for $M$.

There are three main parameters to optimize in this problem: running time, number of maximal tests in the suite $E$, and number of symbols. The last two objectives are important because a suite may be used on multiple black boxes after its construction, or these black-box systems may be slow to execute. Thus, for some applications it may be worthwhile to develop a slower algorithm that results in smaller suites. We adopt the convention that suites produced by conformance testing algorithms are returned by listing their maximal tests. In these circumstances, the time cost of such algorithms is trivially bounded by the total length of the suites they construct.

Methods developed to solve Problem 1 can be understood as modifications of the first technique, the W-method 25, 27. Despite the notable experimental improvements (e.g., 6, 22]), the worst-case analysis of newer techniques does not improve that of the original algorithm, as the W-method is optimal in the worst case 27.

We discuss now the complexity of the W-method. Fix a reduced specification machine $M$. We call the parameter $e:=k-\left|S_{M}\right|$ the number of extra states. This is a central variable in conformance testing, as it measures the uncertainty about the black-box under test. The problem only is meaningful when $e \geq 0$. The number of (maximal) tests produced by the W -method is $O\left(\left|S_{M}\right|^{2}\left|I_{M}\right|^{e+1}\right)$, and the total number of symbols, as well as its time cost, are given by $O\left(\left|S_{M}\right|^{2} k\left|I_{M}\right|^{e+1}\right)$. Some insight on these bounds can be gained from the general structure of conformance testing methods. In most of them, the suite $E$ is built in three stages. First, one constructs a state-cover $V$ of $M$ - i.e., a set containing a word $\alpha$ with $\delta_{M}(\alpha)=s$ for each $s \in S_{M}$. Afterwards, one appends to $V$ the so-called traversal set $\left(I_{M}\right)^{e+1}$, of arbitrary words of length $e+1$. This addition is unavoidable and it is responsible for the exponential factor in the previous bounds. Finally, some distinguishing suffixes are appended to each word in $V \cdot\left(I_{M}\right)^{e+1}$. Improvements over the W-method usually revolve around modifications of this last step.

## 3 Problem Statement

In this section we introduce the restricted conformance testing problem, which is the main subject of this text. As before, let $M$ be a Mealy machine representing a specification for a black box $B$. Let $A$ be an NFA over $I_{M}$ representing the context in which $B$ operates. We consider the extension of the conformance testing problem where it is not possible to apply arbitrary tests to $B$, but only those sequences in $\mathcal{L}_{A}$ can be used instead. Furthermore, now we do not ask whether $M$ and $B$ are equivalent, but just whether they respond equally to sequences in $\mathcal{L}_{A}$. That is, whether $M \sim_{\mathcal{L}_{A}} B$.

A test suite for $M$ in the context of $\mathcal{L}_{A}$ is a finite prefix-closed set $E \subseteq \mathcal{L}_{A}$. Analogously to before, we say that $E$ is $k$-complete (in the context of $\mathcal{L}_{A}$ ) if whenever $M \sim_{E} N$ for some $N \in \Im_{k}$, it also holds that $M \sim_{\mathcal{L}_{A}} N$. Sometimes we will drop the phrase "in the context of $\mathcal{L}_{A}$ ", and simply say that $E$ is $k$-complete when $\mathcal{L}_{A}$ is implied and there is no ambiguity. We study the following problem:

Problem 2 (Restricted conformance testing). Provided with a Mealy machine $M$, an NFA $A$ over $I_{M}$, and some $k \in \mathbb{N}$, compute a $k$-complete suite $E \subseteq \mathcal{L}_{A}$ for $M$ in the context of $\mathcal{L}_{A}$.

As mentioned during the introduction, our motivation for this task lies in the gray-box testing problem were the component under test has observable interfaces, but uncontrollable inputs. During this paper, we focus in the following particular case.


Fig. 1. A cascade composition of Mealy machines

### 3.1 Testing of The Tail Component

A cascade composition $T \circ H$ of two Mealy machines, $T$ and $H$ consists in a one-way sequential connection of both, where the head $H$ processes external inputs and the tail $T$ reacts to $H$ 's outputs (Figure 3 ). In this setting, $T$ can only respond to sequences belonging to $\operatorname{Out}(H)$. An NFA representing this language is easily obtained by "removing" the input symbols from $H$ 's transitions, as shown in Figure 2 [12]. This is called the image automaton of $H, \operatorname{Im}(H)$. This construction shows a straight-forward reduction of the following task to Problem 2.

Problem 3 (Tail component testing). Given a cascade of Mealy machines $T \circ H$, and some $k \in \mathbb{N}$, compute a $k$-complete suite $E \subseteq \operatorname{Out}(H)$ for $T$ in the context of $\operatorname{Out}(H)$.


Fig. 2. Construction of the image automaton for a Mealy machine.

To simplify the discussion we will use this particular case of component testing to motivate our main problem. However, more general forms of component testing were interfaces are observable can also be addressed via Problem 2, as there are polynomial reductions transforming this scenarios into cascade compositions [28, 14]. Now we give a brief overview existing solutions for the Tail Testing problem.

Baseline Solution: Testing of The Composite Machine Given a cascade $T \circ H$, and a bound $k$, one can use existing black-box testing methods to solve Problem 3 in the following way. First, a Mealy machine $P$ representing the whole composition can be obtained via a simple product construction (9]. Here, $I_{P}=I_{H}, O_{P}=O_{H} \times O_{T}$, and $S_{P} \subseteq S_{H} \times S_{T}$. Afterwards, one can apply any existing conformance testing method to obtain a $\left|S_{H}\right| k$-complete suite $E$ for $P$. Finally, computing the image $\lambda_{H}(E)$ of $E$ through $H$ we obtain a $k$ complete suite for $T$ in the context of $\operatorname{Out}(H)$. Taking into account the bounds in section 2.1, the complexity of this approach is $O\left(\left|S_{H}\right|^{3}\left|S_{T}\right|^{2} k\left|I_{M}\right|^{e_{P}+1}\right)$, where $e_{P}:=k\left|S_{H}\right|-\left|S_{P}\right|$. We note that even when $k=\left|S_{T}\right|$ and the original problem presents no extra states, $\left|S_{P}\right|$ can be much smaller than $\left|S_{H}\right|\left|S_{T}\right|$, yielding a large $e_{P}$ and making this approach impractical. We refer to this problem as the blow-up of extra states.

Related Work To the date there are two main approaches proposed for the Tail Testing problem which aim to overcome the blow-up of extra states of the previous method. They are the State-Counting method 17] and a more recent SAT-based technique [19]. Each one of these techniques encounter important issues in their complexity analyses, however, and there is a lack of experimental data about their performance outside very small examples (compositions not reaching ten states in total).

The State-Counting method 17 gives sufficient conditions for complete fault detection in presence of input restrictions. In order to apply these conditions to Problem 3, one has to employ the Kim-Newborn construction 12, as described in [18. This involves constructing a so-called "incompletely specified machine" $P^{\prime}$, via a product of $T$ and the determinization of the image automaton $\operatorname{Im}(H)$. The resulting size of $P^{\prime}$ is $\left|S_{T}\right| 2^{\left|S_{H}\right|}$ in the worst case. This machine $P^{\prime}$ is used later as the specification model to produce a $k$-complete suite. The drawback of this analysis is, however, that this model $P^{\prime}$ can be exponentially bigger than the composite machine $P$ in the baseline method. This potentially yields exponentially larger suites with exponentially longer tests.

The SAT-based approach in 19 constructs a $k$-complete suite $E$ for $T$ in an iterative way, asking a SAT solver whether there is some $T^{\prime} \in \Im_{k}$ with $T \sim_{E} T^{\prime}$ but $T \propto_{\operatorname{Out}(H)} T^{\prime}$. If the answer is negative, $E$ is already $k$-complete. Otherwise, a suitable distinguishing sequence for $T$ and $T^{\prime}$ is added to $E$. This technique has the potential for producing small suites, but the drawback of having to perform a possibly expensive SAT call for the computation of each individual test, whose cost scales exponentially with $\left|S_{H}\right|,\left|S_{T}\right|, k$ and $|E|$.

## 4 Theoretical Analysis

During this section $M$ denotes a specification Mealy machine, $A$ a context NFA over $I_{M}$, and $E$ an unspecified test suite $E \subseteq \mathcal{L}_{A}$. Lastly, we consider a reflexive binary relation $\sqsubseteq$ over $S_{A}$ which under-approximates language containment. That is, $a \sqsubseteq b$ implies $\mathcal{L}_{A}(a) \subseteq \mathcal{L}_{A}(b)$ for all $a, b \in S_{A}$. The goal of this section
is to give sufficient conditions for $k$-completeness of the suite $E$. These, in turn, will provide the correctness guarantees for our proposed algorithms (Section 5 ).

Our sufficient conditions build upon those in the State-Counting method 17, and can be seen as a generalization of them. Informally, the main difference is that the State-Counting method only relates to the case where $A$ is deterministic.

### 4.1 Product of a Mealy Machine with an NFA

Suppose we want to study the observable behaviours of $M$ after the application of a test $\alpha \in \mathcal{L}_{A}$. Here, not only is it relevant to know the state $\delta_{M}(\alpha)$, but also the set of possible context states $a \in \Delta_{A}(\alpha)$. This is because these states $a$ determine which suffixes that can extend the test $\alpha$. Thus, in our setting, state pairs $(s, a) \in S_{M} \times S_{A}$ play a major role.

The product transition function is the map given by $\Delta_{M \times A}((s, a), \alpha)=$ $\left\{\delta_{M}(s, \alpha)\right\} \times \Delta_{A}(a, \alpha)$, for any $(s, a) \in S_{M} \times S_{A}, \alpha \in I_{M}^{*}$. Additionally, given $\alpha \in I_{M}^{*}$, we write $\Delta_{S \times A}(\alpha)$ to denote $\Delta_{S \times A}\left(\left(r_{M}, r_{A}\right), \alpha\right)$.


Fig. 3. Representation of the product of a Mealy machine and a context NFA.

Informally, the semantics of the product $M \times A$ equipped with $\Delta_{M \times A}$ are those of a universally branching machine. Given an input word $\alpha$, an execution of this product consists on multiple parallel runs, each one being the product of a single run of $A$ on $\alpha$ with the deterministic run of $M$ on this sequence. This notion of product of a Mealy machine with an NFA is explored in greater detail in 14 .

In a state pair $(s, a) \in S_{M} \times S_{A}$, the state $s$ of $M$ is responsible for the input/output behaviour, while $a$ represents the input sequences that are nonblocking at this point. Two pairs $(s, a),(t, b) \in S_{M} \times S_{A}$ are distinguishable or incompatible, denoted $(s, a) \nsim(t, b)$ if $\lambda_{M}(s, \alpha) \neq \lambda_{M}(t, \alpha)$ for some sequence $\alpha$ available in both $(s, a)$ and $(t, b)$, i.e., $\alpha \in \mathcal{L}_{A}(a) \cap \mathcal{L}_{A}(b)$. In this situation we say that $\alpha$ witnesses $(s, a) \nsim(t, b)$, written $\alpha \models(s, a) \nsim(t, b)$. Two pairs are equivalent, denoted $(s, a) \cong(t, b)$, if, in addition to being compatible, it holds $a=b$.

In the following result, we bound the length of shortest distinguishing sequences for state-pairs in $S_{M} \times S_{A}$ (see Appendix A for the proof):

Theorem 1. Let $M$ be a Mealy machine and let $A$ be an NFA over $I_{M}$. Let $s, t \in S_{M} b, a \in S_{A}$. Suppose that $(s, a) \nsim(t, b)$ as well as $a \sqsupseteq b$. Then there exists some $\alpha \in \mathcal{L}_{A}(b)$ satisfying both $\alpha \models(s, a) \nsim(t, b)$ and $|\alpha| \leq\left|S_{M}\right|\left|S_{A}\right|$.

### 4.2 Context Tree

During our discussions we need to consider the "unrolling" of the context automaton $A$ on various words. We formalize this notion in the following definition. The context tree is the set $\Gamma \subseteq \mathcal{L}_{A} \times S_{A}$ consisting of the pairs $a / \alpha$, where $a \in \Delta_{A}(\alpha)$. The elements $a / \alpha$ of the testing tree are called nodes. A node $a / \alpha$ is read as " $a$ at $\alpha$ ", and represents a point during an execution of $A$. Given a set of sequences $D \subseteq \mathcal{L}_{A}$, we put $\Gamma(D)$ for the nodes ${ }^{a} / \alpha \in \Gamma$ with $\alpha \in D$. We say that a node ${ }^{b} / \beta$ precedes another one $a / \alpha$, written $b / \beta \preceq a / \alpha$, if $\alpha=\beta \gamma$ and $a \in \Delta_{A}(b, \gamma)$, for some $\gamma$.

Two tests $\alpha, \beta \in E$ are called $E$-separable, denoted $\alpha \#_{E} \beta$, if there is a suffix $\gamma$ satisfying $\alpha \gamma, \beta \gamma \in E$ and $\lambda_{M}\left(\delta_{M}(\alpha), \gamma\right) \neq \lambda_{M}\left(\delta_{M}(\beta), \gamma\right)$. This notion of separability has been used in classical conformance testing [20], and learning (called "apartness") [26]. The following result gives justification for it.

Lemma 1. Suppose that $\alpha \#_{E} \beta$ for two tests $\alpha, \beta \in E$. Then $\delta_{N}(\alpha) \neq \delta_{N}(\beta)$ for any $N \in \Im$ satisfying $M \sim_{E} N$.

Each node $a / \alpha \in \Gamma$ corresponds naturally to a location $\left(\delta_{M}(\alpha), a\right) \in S_{M \times A}$. Given a set of nodes $R \subseteq \Gamma(E)$, we say that $E$ is incompatibility-preserving with respect to (w.r.t.) $R$ if for any $a / \alpha, b / \beta \in C$ with $\left(\delta_{M}(\alpha), a\right) \nsim\left(\delta_{M}(\beta), b\right)$ it holds $\alpha \#_{E} \beta$.

### 4.3 Rankings and Basic Proof of Completeness

During this section we prove a weaker version of our main result where the central arguments of the full proof are showcased. A node ranking is a sequence $\left(a_{j} / \alpha_{j}\right)_{j=1}^{m} \subseteq \Gamma$ of nodes where $\alpha_{1}<\cdots<\alpha_{m}$. We call a ranking flat if $a_{1}=$ $\cdots=a_{m}$, and monotonous if $a_{1} \sqsupseteq \cdots \sqsupseteq a_{m}$. We will be loose with the use of notation and treat rankings as sets when convenient, instead of sequences. We write $R \preceq a / \alpha$ for a ranking $R$ whenever $b / \beta \preceq a / \alpha$ holds for all elements ${ }^{b} / \beta \in R$.

We say that a node $a / \alpha$ is $k$-saturated if there is a monotonous ranking $R \subseteq \Gamma(E)$ with $|R|=k$, where $b / \beta \preceq a / \alpha$ for all $b / \beta \in C$, and $E$ is incompatibilitypreserving w.r.t. $R$. If all the nodes $a^{\prime} / \alpha$ with $a^{\prime} \in \Delta_{A}(\alpha)$ are $k$-saturated, then we say that the sequence $\alpha$ is $k$-saturated itself.

Theorem 2. Suppose that all tests $\alpha \in \mathcal{L}_{A} \backslash E$ have a prefix $\beta \in E$ which is $(k+1)$-saturated. Then $E$ is $k$-complete.

Proof. The proof follows an argument of infinite descent. The idea is that given a test $\alpha \in \mathcal{L}_{A} \backslash E$ which detects a fault not covered by $E$, another strictly shorter sequence $\alpha^{\prime}$ with the same properties can be found. As decreasing sequences of natural numbers are necessarily finite, this scenario is impossible and full fault detection by $E$ is guaranteed. The central part of the "shrinking" argument is that whenever a sufficiently large ranking $R$ can be found throughout a test $\alpha$, then this sequence necessarily follows a "lasso"-like path in the product $M \times A$ and some central portion of $\alpha$ can be removed.

We proceed by contradiction. Let $N \in \Im_{k}$ be a machine satisfying both $M \sim_{E} N$ and $M{\nsim \mathcal{L}_{A}} N$. Let $\alpha \in \mathcal{L}_{A} \backslash E$ be a shortest test distinguishing $M$ and $N$. We show that it is possible to build an even shorter sequence $\alpha^{\prime}$ that also distinguishes $M$ and $N$. Let $\beta \in E$ be a $(k+1)$-saturated prefix of $\alpha$, and let $\gamma$ be the suffix satisfying $\beta \gamma=\alpha$. As $\alpha \in \mathcal{L}_{A}$, it must be that $\gamma \in \mathcal{L}_{A}(b)$ for some $b \in \Delta_{A}(\beta)$. The node $b / \beta$ is $(k+1)$-saturated, so there is some monotonous ranking $R \subseteq \Gamma(E)$ witnessing this property. Let $R=\left(c_{j} / \varphi_{j}\right)_{j=1}^{k+1}$. As $\left|S_{N}\right| \leq k$, by the pigeonhole principle there must be two indices $x<y$ for which $\delta_{N}\left(\varphi_{x}\right)=\delta_{N}\left(\varphi_{y}\right)$. Let $\omega$ be the suffix satisfying $\varphi_{y} \omega=\beta$, and $\varphi_{y} \omega \gamma=\alpha$. Let $\alpha^{\prime}:=\varphi_{x} \omega \gamma$ The following statements hold true:
$\operatorname{Claim}(I) . \alpha^{\prime} \in \mathcal{L}_{A}$.
First, note that $\omega \gamma \in \mathcal{L}_{A}\left(c_{y}\right)$. Indeed, this follows from $\left(\varphi_{y}, c_{y}\right) \preceq b / \beta$ together with $\gamma \in \mathcal{L}_{A}(b)$. As $c_{x} \sqsupseteq c_{y}$, it also holds that $\omega \gamma \in \mathcal{L}_{A}\left(c_{x}\right)$. This, in conjunction with $c_{x} \in \Delta_{A}\left(\varphi_{x}\right)$, shows the claim.

Claim. $\lambda_{M}\left(\delta_{M}\left(\varphi_{y}\right), \omega \gamma\right) \neq \lambda_{N}\left(\delta_{N}\left(\varphi_{y}\right), \omega \gamma\right)$.
The fact that $M \sim_{E} N$ and $\varphi_{y} \in E$, implies $\lambda_{M}\left(\varphi_{y}\right)=\lambda_{N}\left(\varphi_{y}\right)$. However, we know that $\lambda_{M}(\alpha) \neq \lambda_{N}(\alpha)$, and $\alpha=\varphi_{y} \omega \gamma$, so the claim follows.
$C l a i m ~(I I I) . \lambda_{N}\left(\delta_{N}\left(\varphi_{x}\right), \omega \gamma\right)=\lambda_{N}\left(\delta_{N}\left(\varphi_{y}\right), \omega \gamma\right)$.
This is straight-forward, as $\delta_{N}\left(\varphi_{x}\right)=\delta_{N}\left(\varphi_{y}\right)$.
$\operatorname{Claim}(I V) . \lambda_{M}\left(\delta_{M}\left(\varphi_{x}\right), \omega \gamma\right)=\lambda_{M}\left(\delta_{M}\left(\varphi_{y}\right), \omega \gamma\right)$.
Suppose that $\left(\delta_{M}\left(\varphi_{x}\right), c_{x}\right) \nsim\left(\delta_{M}\left(\varphi_{y}\right), c_{y}\right)$. As $R$ is a ranking witnessing that ${ }^{b} / \beta$ is $(k+1)$-saturated, $E$ is incompatibility-preserving w.r.t. $R$. Thus, $\varphi_{x} \#_{E} \varphi_{y}$ follows. However, by Lemma 1 this contradicts the fact that $\delta_{N}\left(\varphi_{x}\right)=\delta_{N}\left(\varphi_{y}\right)$ while at the same time $M \sim_{E} N$. Hence, $\left(\delta_{M}\left(\varphi_{x}\right), c_{x}\right) \sim\left(\delta_{M}\left(\varphi_{y}\right), c_{y}\right)$ must hold. This implies the statement, because $\omega \gamma \in \mathcal{L}_{A}\left(c_{x}\right) \cap \mathcal{L}_{A}\left(c_{y}\right)$, as evidenced during the first claim.

These four claims put together show that $\alpha^{\prime}$ belongs to $\mathcal{L}_{A}$, while also distinguishing $M$ and $N$. However $\left|\alpha^{\prime}\right|<|\alpha|$, contradicting our initial choice of $\alpha$. Thus, no machine $N \in \Im_{k}$ can satisfy $M \sim_{E} N$ and $M \nsim_{\mathcal{L}_{A}} N$ at the same time. This completes the proof of our theorem.

### 4.4 Cores and Covers

Analogously to classical conformance testing algorithms, our proposed methods rely on the initial construction of "cover" of relevant locations. For this we use a notion of core equivalent to the one appearing in 17 .

We say that a set $V \subseteq \mathcal{L}_{A}$ is well-founded if $\epsilon \in V$. Let $V$ be a wellfounded set. For a word $\alpha \in \mathcal{L}_{A}$, we define $|\alpha|_{v}$ as the length of the shortest suffix $\gamma$ satisfying $\beta \gamma=\alpha$, for some $\beta \in V$. Given words $\alpha, \beta$, we write $\beta \leq_{v} \alpha$ if $\beta \leq \alpha$ and additionally $\beta<\gamma<\alpha$ holds for no sequence $\gamma \in V$. Intuitively, this means that $\beta$ lies along the shortest path from $V$ to $\alpha$. It is straightforward to see that $\leq_{v}$ constitutes a partial order over $\mathcal{L}_{A}$. Finally, we put ${ }^{b} / \beta \preceq_{v} a / \alpha$ for a pair of nodes if $\beta \leq_{V} \alpha$, in addition to $b / \beta \preceq a / \alpha$. Given a ranking $R$, we define $R \preceq_{V} a / \alpha$ analogously as before.

We call a set of locations $Q \subseteq S_{M \times A}$ a core, if for all $(s, a) \in S_{M \times A}$ there is some $(t, b) \in Q$ with $(s, a) \sim(t, b)$ and $b \sqsupseteq a$. A core cover is a well-founded set $V \subseteq \mathcal{L}_{A}$ for which the set $\left\{(s, a) \mid \exists \alpha \in V,(s, a) \in \Delta_{S \times A}(\alpha)\right\}$ is a core.

### 4.5 Certificates and Main Condition for Completeness

Here we give our main sufficient condition for suite completeness. This condition is enforced constructively by our proposed algorithms (Section 5), ensuring that they produce $k$-complete suites, as required. For the remainder of the section, we fix a core $Q \subseteq S_{M \times A}$ and a corresponding cover $V \subseteq E$, in addition to $M, A, E, \sqsubseteq$, which were set beforehand.

Given a node ranking $R \subseteq \Gamma$, a basis for $R$ is another set of nodes $B \subseteq \Gamma(V)$ satisfying the following two properties: (1) Nodes in $B$ correspond to pair-wise incompatible locations. That is, $\left(\delta_{M}(\alpha), a\right) \nsim\left(\delta_{M}(\beta), b\right)$ for all nodes $a / \alpha, b / \beta \in$ $B$. (2) Whenever $\left(\delta_{M}(\alpha), a\right) \sim\left(\delta_{M}(\beta), b\right)$ holds for some $a / \alpha \in B, \beta / b \in C$, it follows that $a \sqsupseteq b$. Intuitively, this means that $B$ represents more "testable" locations than $R$.

A redundancy certificate for a node $a / \alpha$ is a pair $(R, B)$ where $R \subseteq \Gamma(E \backslash$ $V)$ is a monotonous ranking satisfying $R \preceq_{v} a / \alpha$, and $B \subseteq \Gamma(V)$ is a basis for $R$. Note that according to this definition $R$ and $B$ are disjoint. Analogously to rankings, certificate is flat if all nodes in $R \cup B$ correspond to the same state $a \in S_{A}$. We say that a node $a / \alpha \in \Gamma$ is $k$-redundant if there is some redundancy certificate $(R, B)$ for the node $a / \alpha$ which satisfies $|R|+|B|=k$ and $E$ is incompatibility preserving w.r.t. $R \cup B$. Analogously as with $k$-saturated sequences, we say that a test $\alpha \in \mathcal{L}_{A}$ is $k$-redundant if all the nodes $a / \alpha$, where $a \in \Delta_{A}(\alpha)$, are $k$-redundant themselves.

Theorem 3. Suppose that all tests $\alpha \in \mathcal{L}_{A} \backslash E$ have a $(k+1)$-redundant prefix $\beta \in E$, satisfying $\beta \leq_{v} \alpha$. Then $E$ is $k$-complete.

The proof is similar to the one of Theorem 2. The main argument relies on showing that distinguishing sequences $\alpha$ outside of $E$ can be "shrunk" as well. The two main differences are that now the relevant measure of size is $|\alpha|_{v}$ rather than $|\alpha|$, and that in the combinatorial arguments we exploit the sizes of
certificates $(R, B)$, rather than those of rankings $R$, as before. The full proof can be found at Appendix B

## 5 Proposed Algorithms

In this section we give high-level descriptions of two algorithms for the restricted conformance problem. Let $M$ be an specification machine and $A$ a context automaton, as before. We present two algorithms for the restricted conformance testing problem, dubbed Simple and Complex, which use the theory developed so far. Both procedures mainly differ in whether they attempt to exploit the language inclusion relation over $S_{A}$.

### 5.1 Simple Variant

Our procedure Simple uses a generalization of the concept of harmonized identifiers adapted to our context. A family of harmonized identifiers is given by a set of words $W_{(s, a)}$ for each location $(s, a) \in S_{M \times A}$ satisfying (1) $W_{(s, a)} \subseteq \mathcal{L}_{A}(a)$, (2) whenever $(s, a) \nsim(t, a)$ for some $(s, a),(t, a) \in S_{M \times A}$, some $\alpha \in W_{(s, a)} \cap$ $W_{(t, a)}$ witnesses $(s, a) \nsim(t, a)$. Note that the sets $W_{(s, a)}$ only need to distinguish $(s, a)$ from other locations corresponding to the same context state $a$.

Algorithm 1 shows the basic structure of SIMPLE. The algorithm constructs a $k$-complete suite $E$ by successively adding various sequences to it. We assume $E$ to be prefix-closed throughout the exposition. Hence, whenever we include a test $\alpha$ in $E$, all its prefixes are implicitly added as well. We initialize the suite $E$ to a cover $V$ of some core $Q$ (line 3 ). The routine $\operatorname{WEAKCore}()$ simply selects one location $(s, a)$ from each equivalence class $S_{M \times A} / \cong$, and $\operatorname{Cover}(Q)$ explores $\mathcal{L}_{A}$ in a breath-first fashion until all locations in $Q$ have been visited. Afterwards, we compute a family of harmonized identifiers $W_{(s, a)}$, and enlarge $E$ by appending them to suitable sequences $\alpha \in V$ (line5). Finally we expand $E$ in a depth-first way starting from each word $\alpha_{v} \in V$ (line 6).

```
Algorithm \(1 \operatorname{Simple}(M, A, k)\)
    Input A specification machine \(M\), context automaton \(A\), and a bound \(k\).
    Output A \(k\)-complete suite \(E\) for \(M\) in the context of \(A\).
    \(Q \leftarrow \mathrm{WEAKCore}()\)
    \(V\), toCvr \(\leftarrow \operatorname{Cover}(Q) \quad \triangleright\) toCvr is a map \(Q \rightarrow V\) where
                                \((s, a) \in \Delta_{M \times A}(t o C v r(s, a))\)
    \(E \leftarrow V\)
    \(\left\{W_{(s, a)}\right\}_{(s, a)} \leftarrow\) family of harmonized identifiers
    for all \((s, a) \in Q\) do \(E \leftarrow E \cup \alpha W_{(s, a)}\), where \(\alpha:=\operatorname{toCvr}(s, a)\)
    for all \(\alpha \in V\) do \(\alpha_{V} \leftarrow \alpha\), and \(\operatorname{Explore}(\epsilon)\)
    return \(E\)
```

The final depth-first exploration carried out in the routine $\operatorname{Explore}(\beta)$, shown in Algorithm 2 The search conducted in a recursive manner starting
from $\alpha_{v}$. This is done by expanding a candidate suffix $\beta$ successively. For this purpose, we examine each possible continuation $\alpha_{v} \beta i$ and determine whether the search space can be pruned at that point. We decide to stop exploring from $\alpha_{v} \beta i$ if the sequence can be made $(k+1)$-redundant by adding suitable distinguishing sequences. This is done a big enough redundancy certificate for each node $a / \alpha_{V} \beta \in$ $\Gamma$ via $\operatorname{SEARCCErts}(\beta)$, and making $E$ incompatibility preserving w.r.t. these certificates in $\operatorname{Exploit} \operatorname{Cert}(R, B)$. We give a more detailed view of those steps.

```
Algorithm \(2 \operatorname{ExplORE}(\beta)\)
    Input a suffix \(\beta\) with \(\alpha_{v} \beta \in \mathcal{L}_{A}\).
    for all inputs \(i \in I_{M}\) with \(\alpha_{v} \beta i \in \mathcal{L}_{A} \backslash V\) do
        Certs \(\leftarrow \operatorname{SearcCerts}(\beta i)\).
        if Certs \(\neq\) false then
            add \(\alpha_{v} \beta i\) to \(E\)
            for all \((R, B) \in\) Certs do ExploitCert \((R, B)\)
        else Explore( \(\beta i\) )
        end if
    end for
    return
```

The function $\operatorname{SearcCerts}(\beta)$, shown in Algorithm 3, attempts to find a redundancy certificate $\left(R_{a}, B_{a}\right)$ satisfying $\left|R_{a}\right|+\left|B_{a}\right|=k+1$ for each node $a / \alpha_{v} \beta \in \Gamma$. If it succeeds, the family of certificates $\left(R_{a}, B_{a}\right)$ is returned. Otherwise, it just returns false. The search of a certificate $\left(R_{a}, B_{a}\right)$ for a node $a / \alpha_{v} \beta$ is divided in two stages. First, a set Rankings of candidate rankings satisfying $R \preceq_{v} a / \alpha_{v} \beta$ is constructed via $\operatorname{BuildRankingS}(\beta, a)$. Afterwards, for each ranking $R \in$ Rankings we find a suitable basis using the routine $\operatorname{Basis}(R)$, and we check whether $|R|+|\operatorname{BASIS}(R)| \geq k+1$.

```
Algorithm 3 SEARChCERTs \((\beta)\)
Input a suffix \(\beta\) with \(\alpha_{v} \beta \in \mathcal{L}_{A}\).
Output a set Certs of redundancy certificates for \(\alpha_{v} \beta\), or false
    Certs \(\leftarrow\}\)
    for all \(a \in \Delta_{A}\left(\alpha_{v} \beta\right)\) do
        Rankings \(\leftarrow \operatorname{BuildRankings}(\beta, a)\)
        if there for some \(R \in\) Rankings with \(|R|+|\operatorname{Basis}(R)| \geq k+1\) then
            add \((R, \operatorname{Basis}(R))\) to Certs
        else return false
        end if
    end for
    return Certs.
```

In this variant, $\operatorname{BuildRankings}(\beta, a)$ builds a family of flat rankings through a linear scanning of the nodes $c / \varphi \preceq_{V} a / \alpha_{V} \beta$. Given a flat ranking $R:=\left(c / \varphi_{i}\right)_{i=1}^{\ell}$, for a fixed $c \in S_{A}$, the method $\operatorname{Basis}(R)$ constructs a basis for $R$ simply by finding all locations of the form $(s, c)$ in the core $Q$. Finally, the function $\operatorname{Exploit} \operatorname{Cert}(R, B)$ is tasked with making $E$ incompatibility preserving w.r.t. a given flat certificate $(R, B)$ by adding several distinguishing sequences to $E$.

```
Algorithm 4 BuildRankings, Basis, ExploitCert (Simple's version)
    1: procedure BuildRankings \((\beta, a)\)
    Input a suffix \(\beta\) with \(\alpha_{v} \beta \in \mathcal{L}_{A}\), and a state \(a \in \Delta_{A}\left(\alpha_{\nu} \beta\right)\)
    Output A set Rankings of constant rankings \(R \preceq_{v} \frac{a}{\alpha_{V} \beta}\).
        Rankings \(\leftarrow\}\)
        initialize empty rankings \(R_{b_{1}}, R_{b_{2}}, \ldots\) for all \(b_{i} \in S_{A}\).
        \(\Omega \leftarrow\) set of nodes \({ }^{c} / \varphi \preceq_{v}{ }^{a} / \alpha_{v} \beta\).
        for all \(j=1,2, \ldots,|\beta|\), and all \(\left(\alpha_{v} \cdot \beta_{\leq j}, b\right) \in \Omega\) do
            append \({ }^{b} / \alpha_{V} \cdot \beta_{j j}\), to \(R_{b}\).
        end for
        Rankings \(\leftarrow\left\{R_{b}\right\}_{b \in S_{A}}\)
    end procedure
    procedure \(\operatorname{BASIS}(R)\)
    Input A flat ranking \(R=\left(c / \varphi_{j}\right)_{j=1}^{\ell} \subseteq \Gamma\), for some \(c \in S_{A}\).
    Output A basis \(B\) for \(R\).
        \(B \leftarrow\}\)
        for all \((s, c) \in Q\), add \((\operatorname{toCvr}(s, c), c)\) to \(B\)
        return \(B\)
    end procedure
    procedure ExploitCert \((R, B)\)
    A flat redundancy certificate \((R, B)\).
        for all \(c / \varphi \in C\) do
            \(s \leftarrow \delta_{M}(\varphi)\)
            add \(\varphi W(s, c)\) to \(E\).
        end for
    end procedure
```


### 5.2 Complex Variant

The basic structure of the method Complex is is largely similar that of Simple. The main difference is that Complex takes an additional parameter $\sqsubseteq$, which is an under under-approximation of language inclusion over $S_{A}$. The goal of Complex is to exploit $\sqsubseteq$ to obtain a possibly more reduced suite than Simple. The detailed description of the algorithm is mostly technical in nature an can be found in Appendix C ADVANced uses $\sqsubseteq$ two main different ways: (1) It uses $\sqsubseteq$ for computing the core $Q$, yielding a possibly smaller initial cover than Simple.
(2) It uses $\sqsubseteq$ to search for non-flat chains and certificates. This potentially allows Advanced to prune the exploration process space earlier than Simple.

The procedure however, shows two main disadvantages with respect to the simpler variant. The first is that searching for general certificates costs more time than searching just for flat ones, as Simple does. The second is that making a suite incompatibility-preserving w.r.t. general certificates requires more involved strategies for adding distinguishing suffixes. Here the idea of using harmonized identifiers does not work, as one needs to distinguish locations $(s, a),(t, b)$ for $a \neq b$, and Complex potentially adds more distinguishing sequences, or longer ones.

### 5.3 Complexity Bounds

In this section we study the complexity of our procedure Simple both in terms of time and sizes of the output suites. We also give the related expressions for Complex. Two notable aspects come out from of this analysis. One is that our methods avoid the addition of exponential-length tests, issue which the StateCounting approach [17 suffered from. The second is that our proposed techniques spend polynomial time in the generation of each test sequence, unlike the SAT-based approach from 19].

Fix $M, A, k$, with $k \geq\left|S_{M}\right|$. First we sketch a bound for the total number of tests in the suite $\operatorname{Simple}(M, A, k)$. Let $n_{(M \times A)}:=\left|S_{M \times A}\right| \cong \mid$. The core $Q$ contains a location $(s, a)$ for each class in $S_{M \times A} \xlongequal[\cong]{ }$. Thus, $|Q| \leq n_{M \times A} \leq\left|S_{M}\right|\left|S_{A}\right|$, and a cover $V$ for $Q$ contains at most $\left|S_{M} \| S_{A}\right|$ words. Now we give a bound the depth of the exploration process carried out in Explore. The following result refers to the scope of Simple. Its proof can be found at Appendix D

Theorem 4. Fix $\alpha_{V} \in V$. Let $\beta$ be a suffix with $\alpha_{V} \beta \in \mathcal{L}_{A}$ and $|\beta|=k\left|S_{A}\right|-$ $n_{M \times A}+1$ Then the method $\operatorname{SearchCerts}(\beta)$ does not return false.

Let $e:=k\left|S_{A}\right|-n_{M \times A}$. The parameter $e$ plays a similar role in this analysis to the number of extra states in traditional conformance testing. Last result shows that the Explore in the worst case may add possible suffixes $\beta$ of size $e+1$ to each word $\alpha_{v} \in V$. This yields potentially $\left|S_{M}\left\|S_{A}\right\| I_{M}\right|^{e+1}$ sequences of the form $\alpha_{v} \beta$. For each of these, Simple appends appends potentially $\left|S_{A}\right|$ identifiers $W_{(s, a)}$, either during its initial phase or during ExploitCerts. This yields an upper bound of $\left|S_{A}\right|^{2}\left|S_{M}\right|^{2}\left|S_{M}\right|\left|I_{M}\right|^{e+1} \mid$ tests in the suite returned by Simple.

To obtain the total number of symbols produced by Simple we multiply last bound by the maximum size of a test in the suite. Without loss of generality, tests generated in Simple are of the form $\alpha_{v} \beta \gamma$, where $\alpha_{v}$ belongs to the cover $V, \beta$ is an arbitrary suffix with $|\beta| \leq e+1$, and $\gamma$ is a distinguishing sequence belonging to some haromonized identifier $W_{(s, a)}$. Clearly, $\left|\alpha_{v}\right| \leq\left|S_{A}\right|\left|S_{M}\right|$, and using Theorem 1 yields $\gamma \leq\left|S_{A}\right|\left|S_{M}\right|$ as well. Putting everything together we get $\left|\alpha_{v} \beta \gamma\right| \leq 3\left|S_{A}\right| k$. This gives us a bound expression of $O\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{2}\left|I_{M}\right|^{e+1} \mid\right.$ symbols generated in Simple.

We note that the bounds obtained for Simple are optimal, in the sense that whenever $A$ is the universal NFA with one state, we recover the bounds for the W -method, discussed in Section 2.1. The time cost of analysis of Simple can be gotten from examining the routines SearchCerts and ExploitCert. This can be seen in more detail in Appendix E. The resulting time cost is $O\left(\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{3}+\left|S_{A}\right|^{4}\left|S_{M}\right|\right) e\left|I_{M}\right|^{e+1}\right)$.

For completeness sake we briefly discuss the complexity analysis of ComPlex. The bounds for number of tests and symbols obtained for Simple also apply for this second variant following similar arguments. The time-cost of the procedure is covered in Appendix E, and is given by $O\left(\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{3}+\right.\right.$ $\left.\left.\left|S_{A}\right|^{5}\left|S_{M}\right|\right) e\left|I_{M}\right|^{e+1}\right)$.

## 6 Experimental Results

Our proposed methods were motivated by the task of testing a component with observable interfaces and non-controllable inputs. During our experiments, we evaluated our techniques on the problem of testing the tail $T$ of a cascade composition $T \circ H$ (Problem 3). For this, we use the reduction described in Section 3.1, which transforms the head $H$ into a suitable NFA $A$. We aim to answer the following questions: (1) How do our techniques compare against the baseline method presented in Section 3.1. (2) How do the sizes of the component machines and the number of extra states influence our methods? Finally, the theory developed in Section 4 allows for a natural application of approximate techniques for NFA reduction and language-inclusion. Hence, our last question is: (3) what kind of impact do those strategies have? We describe now our experimental setup. Our benchmarks consist of randomly constructed cascades of Mealy machines, formed by a head $H$, and a tail $T$, where $O_{H}=I_{T}$. We say a cascade is of size $n \times m$ if $\left|S_{H}\right|=n$ and $\left|S_{T}\right|=m$. To construct the random benchmarks, we utilized the generator in FSMLib [21], which produces reduced connected Mealy machines with given alphabet sizes and number of states. All experiments were run on an Intel Core i5-6200U ( 2.30 GHz ) machine with a limit of 4 GB RAM memory, and a time limit of 3 minutes

In order to answer the first question, we implemented Simple (Section 5.1) and compared it against the testing of the composite machine described in Section 3.1. To represent this baseline, we applied the H-method 7 on the composite machine $P$, using the implementation provided by FSMLib.

In Figure 4, we compare total numbers of symbols and execution times for Simple and the baseline method. For each tail size $\left|S_{T}\right|=2,4,6,8,10$, 12, we generated one hundred cascades where $\left|S_{H}\right|=5$ and all alphabets were of size 4. We considered no extra states in these experiments. That is, we aimed for $k$-complete suites for $T$, where $k=\left|S_{T}\right|$. Solid lines in our graphs represent median quantities, and areas around those lines are enclosed by the 25 -th and 75 -th percentiles of their respective metrics. We conclude that our proposed method, Simple, greatly outperforms the testing of the composite machine in both selected criteria. The main problem the baseline method encountered was


Fig. 4. Comparison between Simple and the testing of the composite machine.
the space limitation. Already with cascades of size $5 \times 10,31 \%$ of the experiments ran out of memory. The root cause of this was the blow-up of extra states, discussed in Section 3.1 Among the $5 \times 12$ benchmarks, the amount of extra states considered by the baseline was bigger than 9 a $27 \%$ of the times.

In order to study the potential benefits of NFA reduction and languageinclusion techniques, we implemented an additional algorithm representing our best attempt at the gray-box testing problem. Here, first we optimize $A$ 's with the approximate method implemented in the tool Reduce [4]. Afterwards, we compute the so-called "look-ahead forward direct simulation relation", introduced in 4], which gives us an under-approximation $\sqsubseteq$ of language inclusion over $S_{A}$. If this results in a trivial relation, we fall back to Simple. Otherwise, we try to exploit $\sqsubseteq$ by calling Complex (Section 5.2). We dub this whole procedure Advanced. The amount of look-ahead used in both the tool Reduce and the computation of $\sqsubseteq$ was set to 16 .

To address the rest of our questions, we generated two additional batches of 500 cascade compositions each. In the first, we fixed $\left|S_{H}\right|=20$, and generated 100 benchmarks for each value $\left|S_{T}\right|=10,20,30,40,50$. In the second followed the same process with the roles of $\left|S_{H}\right|$ and $\left|S_{T}\right|$ reversed. In order to obtain automata $A$ where minimization and language-inclusion techniques show interesting behaviour, we fixed $\left|I_{H}\right|=6$, and $\left|I_{T}\right|=\left|O_{T}\right|=3$. Experimentally, for values $\left|I_{H}\right| /\left|I_{T}\right|$ smaller than two, we found those techniques to have no effect on $A$ in the majority of times, while for larger values $A$ is easily found to be universal. This is consistent with the results in (4].

Figure 5 displays the experimental data of Simple and Advanced on this second set of benchmarks, with zero additional states under consideration. We do not include the baseline here, as it yielded out of memory errors already in $80 \%$ of $20 \times 10$ and $10 \times 20$ compositions. The general trend is that Advanced produces much smaller suites than Simple at the cost of a greater execution time. Both aspects of this comparison are more pronounced when $H$ grows than when $T$ does so. We attribute these differences largely to the automata reduction step in AdVANCED. In $90 \%$ of the experiments, the minimization call was responsible


Fig. 5. Performance of our proposed algorithms with respect to the size of the head (upper row) and the tail (lower row).
$73.4 \%$ of ADVANCED's execution time, while on half the experiments this number ascends to $95.5 \%$. It is worth pointing out that despite producing larger suites, Simple was able to complete the vast majority of the experiments ( $927 / 1000$ ) in under a second.

For Figure 6 we ran again a subset of the previous experiments, but considering one addditional extra state. Out of the original 1000, we picked the 600 cascades where head and tail had at most 30 states. Here Advanced outperforms Simple in both time and number of symbols. Moreover, minimization time still accounted for a $67 \%$ of ADVANCED's execution time in half of the occasions. In this case, the initial automata minimization step seems clearly beneficial. The observed effect of the additional state is drastic both with respect to execution times and suite sizes. Nevertheless, this impact is much smaller than what our worst-case analyses predict (Section 5.3). According to those, an additional state could worsen the metrics of both procedures by a factor of $\left|I_{T}\right|^{\left|S_{H}\right|}$. This ascends to around $35 \cdot 10^{8}$ for $\left|I_{T}\right|=3$ and $\left|S_{H}\right|=20$. We note that this blowup is unavoidable for black-box testing techniques. However, the relative increase between Figure 5 and Figure 6 is not nearly as large.

Lastly, to evaluate the effect of the language inclusion relation on our algorithms, we implemented an additional procedure Simple + Reduce, which just


Fig. 6. Performance of our algorithms in the presence of extra states.
calls Simple after the initial NFA reduction. We ran the experiments of Figure 5 and Figure 6 on this method, and compared it against ADVANCED. We note that in about $60 \%$ of the experiments both methods performed the same operations, as the relation $\sqsubseteq$ obtained from $A$ was trivial. For the remaining $40 \%$ of the cases, we computed the ratio of symbols produced by AdVanced to symbols produced by Simple+Reduce. This information is summarized in Table 1 We observe that in $75 \%$ of the times exploiting $\sqsubseteq$ by means of Advanced was either noticeably beneficial or had almost no effects. However, in about $10 \%$ of the cases the impact was clearly negative.

| ADVANCED / SIMPLE + Reduce |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Extra States | $1 \%$ | $10 \%$ | $25 \%$ | $50 \%$ | $75 \%$ | $90 \%$ | $99 \%$ | $100 \%$ |  |
| $k=\left\|S_{M}\right\|$ | 0.047 | 0.377 | 0.761 | 1.0 | 1.045 | 1.301 | 1.971 | 2.973 |  |
| $k=\left\|S_{M}\right\|+1$ | 0.006 | 0.249 | 0.515 | 1.0 | 1.009 | 1.360 | 1.926 | $\infty$ |  |

Table 1. Symbols produced by Advanced over symbols produced by SimPle + Reduce

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## A Proof of Theorem 1

It is clearly enough to show the result for $a=b, \alpha \models(s, a) \nsim(t, b)$ is equivalent to $\alpha \models(s, b) \nsim(t, b)$ whenever $\mathcal{L}_{A}(a) \supseteq \mathcal{L}_{A}(b)$.

For any $s, t \in S_{M}$ and $a \in S_{A}$, we write $s \varkappa_{a} t$ as a shorthand for $(s, a) \nsim$ $(t, a)$. Additionally, we say that $s \propto_{a}^{j} t$ when $\alpha \models s \propto_{a} t$ for some $\alpha$ with $|\alpha| \leq j$. We adopt the convention that $s \sim_{a}^{0} t$ for all $s, t$, $a$. Let $m=\left|S_{M}\right|\left|S_{A}\right|$. We show that $\sim_{a}^{m}$ is the same relation as $\sim_{a}$ for all $a \in S_{A}$. Note that this proves our statement. We proceed by showing various claims. The first ones are straight-forward.

Claim 1. The relation $\sim_{a}^{j+1}$ refines $\sim_{a}^{j}$, written $\sim_{a}^{j} \supseteq \sim_{a}^{j+1}$. This means that $s \sim_{a}^{j+1} t$ implies $s \sim_{a}^{j+1} t$ for all $s, t, a, j$.

Claim 2. The relations $\sim_{a}^{j}, \sim_{a}$ are equivalence relations, and $\sim_{a}=\cap_{j=1}^{\infty} \sim_{a}^{j}$.
Claim 3. Suppose that for some $j \in \mathbb{N}$ it holds that $\sim_{a}^{j}=\sim_{a}^{j+1}$ for all $a \in S_{A}$. Then $\sim_{a}^{j}=\sim_{a}^{k}$ for all $k \geq j$ and all $a \in S_{A}$. To show this claim, suppose that $s \sim_{a}^{j} t$ but $s \nsim_{a}^{j+1} t$, for some $j>0$. Let $i_{1} i_{2} \ldots i_{j+1} \in \mathcal{L}(a)$ be a sequence distinguishing $s$ and $t$. Let $s^{\prime}:=\delta_{M}\left(s, i_{1}\right), t^{\prime}:=\delta_{N}\left(t, i_{1}\right)$, and let $b \in \Delta_{A}\left(a, i_{1}\right)$ be such that $i_{2} \ldots i_{j+1}$ belongs to $\mathcal{L}(b)$. Then $s^{\prime} \varkappa_{b}^{j} t^{\prime}$. Furthermore, it cannot be that $s^{\prime} \varkappa_{b}^{j-1} t^{\prime}$ as well. Otherwise $s \chi_{a}^{j} t$ would follow, contradicting our initial assumption. Hence we have shown that if for some $a \in S_{A}, j>0$ it holds $\sim_{a}^{j} \neq \sim_{a}^{j+1}$ then $\sim_{b}^{j-1} \neq \sim_{b}^{j}$ for some $b \in S_{B}$. This is equivalent to the claim.

Now we can complete the proof of our theorem. For each $j \in \mathbb{N}$ consider the set of equivalence relations $\left\{\sim_{a}^{j}\right\}_{a \in S_{A}}$. Because of Claim 3, we know that at each successive step $j=1,2, \ldots$ at least one relation is refined $\sim_{a}^{j} \supsetneq \sim{ }_{a}^{j+1}$, until $\sim_{a}^{j}=\sim_{a}$ for all $a \in S_{A}$. If $\sim_{a}^{j} \supsetneq \sim_{a}^{j+1}$, then $\sim_{a}^{j+1}$ yields strictly more equivalence classes than $\sim_{a}^{j}$. For each $a$, the relation $\sim_{a}^{j}$ can have at most $\left|S_{M}\right|$ equivalence classes. Thus, the relations $\left\{\sim_{a}^{j}\right\}_{a \in S_{A}}$ can be refined at most $m=\left|S_{M}\right|\left|S_{A}\right|$ times in total. This implies $\sim_{a}^{m}=\sim_{a}$ for all $a$, as we wanted to show.

## B Proof of Theorem 3

We proceed by contradiction as in Theorem 2, We take $N \in \Im_{k}$ satisfying $M \sim_{E} N$ and $M{\nsim \mathcal{L}_{A}} N$, and $\alpha \in \mathcal{L}_{A} \backslash E$ a sequence distinguishing $M$ and $N$ which minimizes $|\alpha|_{v}$. This time we show that another sequence $\alpha^{\prime}$ separating $M$ and $N$ as well, with $\left|\alpha^{\prime}\right|_{v}<|\alpha|_{v}$ can be found. Let $\beta \in E$ be a $(k+1)$-redundant prefix of $\alpha$, with $\beta \leq_{v} \alpha$, and let $\gamma$ be the prefix for which $\alpha=\beta \gamma$. Let $b \in \Delta_{A}(\beta)$ be a state satisfying $\gamma \in \mathcal{L}_{A}(b)$. The node ${ }^{b} / \beta$ is $(k+1)$-redundant, so there is a redundancy certificate $(R, B)$ witnessing this property. Let $R=\left(d_{j} / \zeta_{j}\right)_{j=1}^{\ell}$. As $|R \cup B|=k+1$, by the pigeonhole principle there are $c / \varphi_{1}, \varphi_{2} / c_{2} \in C \cup B$ satisfying $\delta_{N}\left(\varphi_{1}\right)=\delta_{N}\left(\varphi_{2}\right)$. Without loss of generality we can assume that $\left|\varphi_{1}\right|_{v} \leq\left|\varphi_{2}\right|_{v}$. As before, we proceed by giving various claims.
$\operatorname{Claim}(I) .\left(\delta_{M}\left(\varphi_{1}\right), c_{1}\right) \sim\left(\delta_{M}\left(\varphi_{2}\right), c_{2}\right)$.
Otherwise we would have $\varphi_{1} \#_{E} \varphi_{2}$, as $E$ is incompatibility preserving w.r.t. $R \cup B$. But this contradicts $M \sim_{E} N$, proving the claim.

Claim (II). Either one of the following holds. Case 1: $c_{1} / \varphi_{1}=d_{x} / \zeta_{x}, c_{2} / \varphi_{2}=d_{y} / \zeta_{y}$, for some indices $x<y$. Case 2: $\left(\varphi_{1}, c_{1}\right) \in B,\left(\varphi_{2}, c_{2}\right) \in C$
Last claim shows that both $c_{1} / \varphi_{1}, c_{2} / \varphi_{2}$ cannot belong to $B$ at the same time, as it would yield a conflict with the definition of basis. Thus there are two possible scenarios: either (i) both nodes belong to $R$, or (ii) exactly one of them lies in $B$. We show that these correspond to Case 1 and Case 2 in the statement, respectively. We begin by assuming (i). In this situation, we know that $c_{1} / \varphi_{1}=d_{x} / \zeta_{x}$, $c_{2} / \varphi_{2}=d_{y} / \zeta_{y}$, for some $x, y$, and we have to prove $x<y$. As $R \preceq_{v}{ }^{b} / \beta$, it holds that $\zeta_{1} \leq_{v} \beta$, so there is no $\xi \in V$ with $\zeta_{1}<\xi<\beta$. This implies $\zeta_{1}<_{V} \cdots<_{V} \zeta_{\ell}$, and as a consequence $\left|\zeta_{1}\right|_{v}<\cdots<\left|\zeta_{\ell}\right|_{v}$. By assumption $\left|\varphi_{1}\right|_{v} \leq\left|\varphi_{2}\right|_{v}$, so $x<y$ follows. Hence, Case 1 holds. Now we assume (ii) instead. Note that for all $(\zeta, d) \in B$ it holds $\zeta \in V$, so $|\zeta|_{v}=0$. Conversely, for all $(\zeta, d) \in C, \zeta \notin V$, and $|\zeta|_{v}>0$. Again, by assumption $\left|\varphi_{1}\right|_{v} \leq\left|\varphi_{2}\right|_{v}$, implying $c_{1} / \varphi_{1} \in B$ and $c_{2} / \varphi_{2} \in C$, as in Case 2.

For the remainder of the proof we will refer to the cases Case 1 and Case 2 in last claim.

Claim (III). The inequality $\left|\varphi_{1}\right|_{v} \leq\left|\varphi_{2}\right|_{V}$ is strict.
Case 1: In the proof of Claim (II) we showed $\left|\zeta_{1}\right|_{v}<\cdots<\left|\zeta_{\ell}\right|_{v}$. Hence the statement follows. In this situation $\varphi_{1}, \varphi_{2}$ lie among the $\zeta_{j}$ 's, so the ranking of inequalities implies our claim. Case 2: Note that for all $(\zeta, d) \in B$ it holds $\zeta \in V$, so $|\zeta|_{v}=0$. Conversely, for all $(\zeta, d) \in C, \zeta \notin V$, and $|\zeta|_{v}>0$. This shows the claim.

Claim (IV). $c_{1} \sqsupseteq c_{2}$.
Case 1: The statement follows from the definition of monotonous ranking. Case 2: By Claim (III), it holds $\left(\varphi_{1}, c_{1}\right) \in B,\left(\varphi_{2}, c_{2}\right) \in C$. Using the definition of basis and Claim (I), we obtain $c_{1} \sqsupseteq c_{2}$ in this case as well.

Now we are in conditions to build the second distinguishing sequence $\alpha^{\prime}$. By Claim (II) $c_{2} / \varphi_{2} \in C$, so $\varphi_{2} \leq \beta$. Let $\omega$ be the suffix satisfying $\beta=\varphi_{2} \omega$. Then $\alpha=\varphi_{2} \omega \gamma$. We define $\alpha^{\prime}$ as the word $\varphi_{1} \omega \gamma$. They following claims can all be shown exactly as in Theorem 2]s proof: $\operatorname{Claim}(V) . \alpha^{\prime} \in \mathcal{L}_{A}$. Claim (VI). $\lambda_{M}\left(\delta_{M}\left(\varphi_{2}\right), \omega \gamma\right) \neq \lambda_{N}\left(\delta_{N}\left(\varphi_{2}\right), \omega \gamma\right) . \quad \operatorname{Claim}(V I I) . \lambda_{N}\left(\delta_{N}\left(\varphi_{1}\right), \omega \gamma\right)=$ $\lambda_{N}\left(\delta_{N}\left(\varphi_{1}\right), \omega \gamma\right) . \quad$ Claim (VIII). $\lambda_{M}\left(\delta_{M}\left(\varphi_{1}\right), \omega \gamma\right)=\lambda_{M}\left(\delta_{M}\left(\varphi_{2}\right), \omega \gamma\right)$.

Claims (V)-(VII) show that $\alpha^{\prime}$ belongs to $\mathcal{L}_{A}$ and distinguishes $M$ from $N$. All that is left is to prove $\left|\alpha^{\prime}\right|_{v}<|\alpha|_{v}$. Using $|\alpha|_{v}=\varphi_{1}$ and $\alpha^{\prime}=\varphi_{2} \omega \gamma$ we obtain (1) $\left|\alpha^{\prime}\right|_{v} \leq\left|\varphi_{2}\right|_{v}+|\omega \gamma|$. Now we show a similar expression for $|\alpha|_{v}$. As $\left(\varphi_{2}, c_{2}\right) \preceq_{v}{ }^{b} / \beta$ it holds $\varphi_{2} \leq_{v} \beta$. Also, by hypothesis, $\beta \leq_{v} \alpha$. Putting the inequalities together we get $\varphi_{2} \leq_{v} \alpha$. This yields $|\alpha|_{v}=\left|\varphi_{2}\right|_{v}+|\omega \gamma|$. Additionally $\left|\alpha^{\prime}\right|_{v} \leq\left|\varphi_{1}\right|_{v}+|\omega \gamma|$. Comparing the expression for $\alpha$ and $\alpha^{\prime}$ and utilizing Claim (III) gets us $\left|\alpha^{\prime}\right|_{v}<|\alpha|_{v}$. This contradicts our initial choice of $\alpha$ and completes the proof of the theorem.

## C Detailed Description of Complex

Algorithm 5 shows the main structure of Complex. The routine Core() first obtains a core $Q$ from WEAKCORE and afterwards removes each location $(s, a) \in$ $Q$ if there is another one $(t, b) \in Q$ where $(s, a) \sim(t, b)$ and $b \sqsupseteq a$. The second difference is that Complex does not make use of harmonized quasi-identifiers, unlike Simple, but instead relies on a map of distinguishing sequences DistSeqs. This map stores a shortest separating sequence $\alpha \models s \varkappa_{a} t$ for each triple $s, t \in S_{M}, a \in S_{A}$, if it exists, or the empty sequence otherwise. Finally, the last difference is that in Complex no distinguishing sequences are added to the cover $V$ initially. Instead we add these sequences dynamically during the exploration process.

```
Algorithm \(5 \operatorname{Complex}(M, A, \sqsubseteq, k)\)
    Input A specification machine \(M\), context automaton \(A\), an under-approximation
\(\sqsubseteq\) of language containment over \(S_{A} \times S_{A}\), and a bound \(k\).
    Output A \(k\)-complete suite \(E\) for \(M\) in the context of \(A\).
    \(Q \leftarrow \operatorname{Core}()\)
    \(V\), toCvr \(\leftarrow \operatorname{Cover}(Q) \quad \triangleright t o C v r\) is a map \(Q \rightarrow V\) where
        \((s, a) \in \Delta_{M \times A}(\operatorname{toCvr}(s, a))\)
    \(E \leftarrow V\)
    DistSeqs \(\leftarrow \operatorname{map} S_{M} \times S_{M} \times A \rightarrow I_{M}^{*}\) assigning a shortest distinguishing sequence
    \(\alpha=s \nsim_{a} t\) for each \(s, t \in S_{M}, a \in S_{A}\).
    for all \(\alpha \in V\) do \(\alpha_{v} \leftarrow \alpha\), and \(\operatorname{Explore}(\epsilon)\)
    return \(E\)
```

In its final step, Complex performs a depth-first search from each word $\alpha_{V} \in V$, enlarging $E$ along the way. For this, the algorithm relies on the same routines Explore and SearchCerts utilized by Simple. We modify, however the functions BuildRankings, Basis, and ExploitCert. Now we can use $\sqsupseteq$ to produce general monotonous rankings, instead of only flat ones as before. This allows Complex to potentially prune the search space earlier, as it can force shorter sequences to become $(k+1)$-redundant.

Similarly as with Simple, the method BuildRankings $(\beta, a)$ builds a family Rankings of monotonous rankings $R \preceq_{v}\left(\alpha_{v} \beta, a\right)$. It does so by building for each $b \in S_{A}$ a maximum-length ranking $\left(c_{j} / \varphi_{j}\right)_{j=1}^{\ell}$ where $c_{j}=b$. This can be done incrementally by scanning the nodes $c / \alpha_{V} \beta^{\prime} \preceq_{V} a / \alpha_{V} \beta$ for each prefix $\beta^{\prime} \leq \beta$.

Finding a greatest basis $B \subseteq \Gamma(V)$ for a monotonous ranking is, in principle, computationally hard, given that this task can be reduced to a maximal independent set problem. However, if we do not aim for a biggest basis, the task can be carried out with relative efficiency. We propose a greedy approach in Basis for this purpose.

Finally, ExploitCert $(R, B)$ is tasked with making $E$ incompatibility preserving w.r.t. the certificate $(R, B)$ by adding various distinguishing sequences, as before. Following a naive approach involves adding a distinguishing sequence
for each pair $\left(\omega_{1}, b_{1}\right),\left(\omega_{2}, b_{2}\right) \in C \cup B$ where $\left(\delta_{M}\left(\omega_{1}\right), b_{1}\right) \nsim\left(\delta_{M}\left(\omega_{2}\right), b_{2}\right)$. This adds up to potentially $(|C|+|B|)^{2}=(k+1)^{2}$ sequences. However, if one chooses the sequences carefully, it is only needed to distinguish the nodes in $R$ with those in $B$, and the nodes in $B$ among themselves. As $|B| \leq\left|S_{M}\right|$, this brings down the number of separating sequences to at most $(k+1)\left|S_{M}\right|$. We implement this technique in Exploitcert.

## D Proof of Theorem 4

We show that in the body of the main loop SearchCerts (Algorithm3) is able to find a big enough redundancy certificate for all nodes $a / \alpha_{V} \beta$. For each $a \in S_{A}$ let $m_{a}$ be the number of different classes in $S_{M \times A} \cong$ corresponding to locations of the form $(s, a)$. Then $|\beta|=\left(k\left|S_{A}\right|-\sum_{a \in S_{A}} m_{a}\right)+1$. Let $a \in \Delta_{A}\left(\alpha_{V} \beta\right)$. Then there is at least one sequence of nodes $\left(\alpha_{V} \beta_{\leq 1}, b_{1}\right) \preceq\left(\alpha_{V} \beta_{\leq 2}, b_{2}\right) \preceq \cdots \preceq$ $\left(\alpha_{V} \beta, a\right)$. This sequence has length $|B|$, so by the pigeonhole principle at one state $b \in S_{A}$ occurs at least $k-m_{b}+1$ times throughout the succession. Thus, this quantity is a lower bound for the size of the flat ranking $R_{b}$, corresponding to $b$, built in the procedure $\operatorname{BuildRankings}(\beta, a)$. Now, note that $\operatorname{Basis}\left(R_{b}\right)$ returns a basis of size exactly $n_{b}$. Hence, $\left|R_{b}\right|+\left|\operatorname{BASIS}\left(R_{b}\right)\right| \geq k+1$, and the conditional in Algorithm 3, line 4 is true. Our initial choice of $a \in \Delta_{A}\left(\alpha_{V} \beta\right)$ was arbitrary, so this proves that $\operatorname{SEARch} \operatorname{Certs}(\beta)$ does not return false.

## E Time-Cost Analysis for Simple and Complex

For the complexity analysis of Simple, we only need to consider the time spent in the routine Explore. Potentially, this function is called once for each word $\alpha_{v} \beta$, where $\alpha_{v} \in V$, and $|\beta| \leq\left|S_{A}\right| k-n_{M \times A}+1$. During these calls, Explore invokes the SEarchCerts once, on $\beta$, and the method ExploitCert at most $\left|S_{A}\right|$ times: one for each certificate returned by SEARChCerts. We analyze both functions separately.

Inside $\operatorname{SearchCerts}(\beta)$ most of the time is spent calling BuildRank$\operatorname{INGS}(\beta, a)$. In this second function the bulk of the time is invested in building the set $\Omega$ of nodes $c / \varphi \preceq_{v}\left(\alpha_{V} \beta, a\right)$. This can be done by back-propagating the node $\left(\alpha_{V} \beta, a\right)$ throughout all words $\alpha_{V} \beta^{\prime}$ with $\beta^{\prime} \leq \beta$. If one stores $A$ reverse transitions, this takes at most $O\left(\left|S_{A}\right|^{2}|\beta|\right)$ time. BuildRankings is called at most $\left|S_{A}\right|$ times in the a single call of SearchCerts. Hence, $\operatorname{SearchCerts}(\beta)$ takes $O\left(\left|S_{A}\right|^{3}|\beta|\right)=O\left(\left|S_{A}\right|^{3}(e)\right)$ time. The method is called once for each word $\alpha_{V} \beta$, so the total amount of time it uses during Simple is $O\left(\left|S_{A}\right|^{4}\left|S_{M}\right|(e)\left|I_{M}\right|^{e+1}\right)$ time.

The workload inside ExploitCert $(R, B)$ is mainly the result of adding distinguishing suffixes. This method adds at most $|R|\left|S_{M}\right|$ of those to $E$, and each one of these sequences has length bounded by $\left|S_{M}\right|\left|S_{A}\right|$. If $E$ is stored in a tree structure, this can be done in $O\left(|C|\left|S_{A}\right|\left|S_{M}\right|^{2}\right)=O\left(k\left|S_{A} \| S_{M}\right|^{2}\right)$ time. The method is called at most $\left|S_{A}\right|$ times for each word $\alpha_{V} \beta$. Hence, Simple spends at most $O\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{3}(e)\left|I_{M}\right|^{e+1}\right)$ time in ExploitCert. Putting the bounds
for SearchCerts and ExploitCert together gives us that the total time cost of $\operatorname{Simple}(M, A, k)$ is $O\left(\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{3}+\left|S_{A}\right|^{4}\left|S_{M}\right|\right) e\left|I_{M}\right|^{e+1}\right)$. Analogous arguments can be used to obtain the complexity of Complex. The only relevant change here is that there is an additional inner loop in the routine BuILDRankIngs, increasing its cost by a factor of $\left|S_{A}\right|$. This yields a total complexity of $O\left(\left(k\left|S_{A}\right|^{3}\left|S_{M}\right|^{3}+\left|S_{A}\right|^{5}\left|S_{M}\right|\right) e\left|I_{M}\right|^{e+1}\right)$ for $\operatorname{Complex}(M, A, \sqsupseteq, k)$.

```
Algorithm 6 BuildRankings, Basis, ExploitCert (Complex version)
    procedure BuILDRANKINGS \((\beta, a)\)
    Input a suffix \(\beta\) with \(\alpha_{v} \beta \in \mathcal{L}_{A}\), and a state \(a \in \Delta_{A}\left(\alpha_{v} \beta\right)\)
    Output A set Rankings of monotonous rankings \(R \preceq_{v} \frac{a}{\alpha_{v} \beta}\).
        Rankings \(\leftarrow\}\)
        initialize empty rankings \(R_{b_{1}}^{0}, R_{b_{2}}^{0}, \ldots\) for all \(b_{i} \in S_{A}\).
        \(\Omega \leftarrow\) set of nodes \({ }^{c} / \varphi \preceq_{v}{ }^{a} / \alpha_{v} \beta\).
        for all \(j=1,2, \ldots,|\beta|\), and all \(b \in S_{A}\) do
            if \(b / \alpha_{V} \cdot \beta_{\leq j} \in \Omega\) then
                Let \(c \in S_{A}\) be the state \(c \sqsupseteq b\) maximizing \(\left|R_{c}^{j-1}\right|\).
                \(R_{b}^{j}=R_{c}^{j-1} \cup \frac{b}{\alpha_{V} \cdot \beta^{\leq i}}\)
            else \(\quad R_{b}^{j}=R_{b}^{j-1}\).
            end if
        end for
        Rankings \(\leftarrow\left\{R_{b}^{|\beta|}\right\}_{b \in S_{A}}\)
    end procedure
    procedure \(\operatorname{Basis}(R)\)
    Input A monotonous ranking \(R=\left(c_{j} / \varphi_{j}\right)_{j=1}^{\ell} \subseteq \Gamma\).
    Output A basis \(B\) for \(R\).
        \(B \leftarrow\}\)
        \(Q^{\prime} \leftarrow\{ \}\)
        for all \(j=1,2, \ldots, \ell\) and \(s \in S_{M}\) do
            if \(Q^{\prime}\) does not contain any \((t, b)\) with \((t, b) \sqsupseteq\left(s, c_{j}\right)\) then
                Find \((t, b) \in Q\) with \((t, b) \sqsupseteq\left(s, c_{j}\right)\), and add it to \(Q^{\prime}\).
            end if
        end for
        for all \((s, c) \in Q^{\prime}, \operatorname{add}(\operatorname{toCvr}(s, c), c)\) to \(B\)
        return \(B\)
    end procedure
    procedure ExploitCert \((R, B)\)
    A redundancy certificate \((R, B)\), where \(R=\left(c_{j} / \varphi_{j}\right)_{j=1}^{\ell}\).
        for all \(j=1, \ldots, \ell\) and all \((\omega, b) \in B\) do
            \(s \leftarrow \delta_{M}\left(\varphi_{j}\right), t \leftarrow \delta_{M}(\omega)\)
            \(x \leftarrow\) maximum index \(j \leq x \leq \ell\) satisfying \(s{\nsim c_{x}} t\).
            Add \(\varphi_{j} \gamma\) to \(E\), where \(\gamma=\operatorname{DistSeqs}\left(s, t, c_{x}\right)\)
        end for
        for all pairs \(\left(\omega_{1}, b_{1}\right),\left(\omega_{2}, b_{2}\right) \in B\) do
            \(s_{1} \leftarrow \delta_{M}\left(\omega_{1}\right), s_{2} \leftarrow \delta_{M}\left(\omega_{2}\right)\)
            \(x \leftarrow\) maximum index \(1 \leq x \leq \ell\) satisfying \(s_{1}{\nsim c_{x}} s_{2}\).
            Add \(\omega_{1} \gamma, \omega_{2} \gamma\) to \(E\), where \(\gamma=\operatorname{DistSeqs}\left(s_{1}, s_{2}, c_{x}\right)\)
        end for
    end procedure
```

